PROBLEM 3

Note that I always write lim

1) $a_n = \left(1 - \frac{1}{2n}\right)^n$ this is of the typ $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

$$\lim_{n\to\infty}\left(1-\frac{1}{2n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{(-2n)}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{(-2n)}\right)^{n(-2)\cdot\frac{1}{-2}}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{(-2n)}\right)^{-2n}\right]^{-\frac{1}{2}} = e^{-1/2}$$

2) $a_n = \left(\frac{n+3}{n}\right)^{2n}$ same type after I split the fraction

$$\lim_{n\to\infty}\left(\frac{n+3}{n}\right)^{2n} = \lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{2n} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{3}}\right)^{2n\cdot\frac{1}{3}\cdot 3}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{n}{3}}\right)^{n/3}\right]^6 = e^6$$

3) $a_n = \left(\frac{n}{n+1}\right)^n$ same after I add and subtract 1

$$\lim_{n\to\infty}\left(\frac{n+1-1}{n+1}\right)^n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{-(+n+1)}\right)^{n+1-1}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{-(n+1)}\right)^{(n+1-1)(-1)(-1)} = \lim_{n\to\infty}\left(1+\frac{1}{-(n+1)}\right)^{(-(n+1)+1)(-1)}$$

$$= \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{1}{-(n+1)}\right)^{-(n+1)}}_{\to e}\right]^{-1}\cdot\left(\underbrace{1+\frac{1}{-(n+1)}}_{\to 1}\right)^{-1} = e^{-1}$$

Alternative methods for the previous problem

1) $\left(1-\frac{1}{2n}\right)^n = e^{\log\left(1-\frac{1}{2n}\right)^n} = e^{n\log\left(1-\frac{1}{2n}\right)}$

Then compute $\lim_{n\to\infty} n\log\left(1-\frac{1}{2n}\right)$

by using de L'Hospital or taylor polynomials.

<u>L'Hospital</u>: $\lim_{n\to\infty} n\log\left(1-\frac{1}{2n}\right) = \lim_{n\to\infty} \frac{\log\left(1-\frac{1}{2n}\right)}{\frac{1}{n}}$ $\left(\text{"}\frac{0}{0}\text{"}\right)$

$\overset{H}{=} \lim_{n\to\infty} \frac{\frac{1}{1-\frac{1}{2n}}\cdot\frac{1}{2\cancel{n^2}}}{-\frac{1}{\cancel{n^2}}} = \lim_{n\to\infty} -\frac{1}{2}\frac{1}{1-\frac{1}{2n}} = -\frac{1}{2}$

so $\lim_{n\to\infty} e^{n\log\left(1-\frac{1}{2n}\right)} = e^{-1/2}$

<u>Taylor</u> $\log(1+x) = x + o(x)$ for $x\to 0$. $-\frac{1}{2n}\to 0$ for $n\to\infty$

so $\log\left(1+\left(-\frac{1}{2n}\right)\right) = -\frac{1}{2n} + o\left(\frac{1}{n}\right)$

$\lim_{n\to\infty} n\log\left(1-\frac{1}{2n}\right) = \lim_{n\to\infty} n\left(-\frac{1}{2n}+o\left(\frac{1}{n}\right)\right) = \lim_{n\to\infty} -\frac{1}{2}+o(1) = -1/2$

so $\lim e^{n\log\left(1-\frac{1}{2n}\right)} = e^{-1/2}$

2) $\left(\frac{n+3}{n}\right)^{2n} = \left(1+\frac{3}{n}\right)^{2n} = e^{2n\log(1+3/n)}$

Taylor: $\frac{3}{n} \to 0$ for $n \to \infty$ so $\log(1+3/n) = 3/n + o\left(\frac{1}{n}\right)$

$\lim_{n\to\infty} 2n\log\left(1+\frac{3}{n}\right) = \lim_{n\to\infty} 2n\left(\frac{3}{n}+o\left(\frac{1}{n}\right)\right) = \lim_{n\to\infty} 6 + o(1) = 6$

So $\lim_{n\to\infty} e^{2n\log(1+3/n)} = e^6$

3) $\left(\frac{n}{n+1}\right)^{n} = \left(\frac{n+1-1}{n+1}\right)^{n} = \left(1-\frac{1}{n+1}\right)^{n} = e^{n\log\left(1-\frac{1}{n+1}\right)}$

Taylor $-\frac{1}{n+1} \to 0$ for $n\to\infty$ so $\log\left(1-\frac{1}{n+1}\right) = -\frac{1}{n+1} + o\left(\frac{1}{n+1}\right)$

$\lim_{n\to\infty} n\log\left(1-\frac{1}{n+1}\right) = \lim_{n\to\infty} -\frac{n}{n+1} + o\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} -\frac{n}{n+1} + o(1) = -1$

So $\lim_{n\to\infty} e^{n\log\left(1-\frac{1}{n+1}\right)} = e^{-1}$

You can also try l'Hospital.

4) $a_n = \left(\frac{1}{n^2}\right)^{\frac{1}{n}}$ undefined form "0^0"

$$= e^{\frac{1}{n}\log\left(\frac{1}{n^2}\right)} = e^{-\frac{2}{n}\log(n)}$$

$$\lim_{n\to\infty} -\frac{2}{n}\log(n) \overset{H}{=} \lim_{n\to\infty} \frac{-\frac{2}{n}}{1} = 0$$

so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{-\frac{2}{n}\log(n)} = e^0 = 1$

5) $a_n = \sqrt{n}\,\log(n)$ is not undefined

$$\lim_{n\to\infty} \sqrt{n}\,\log(n) = +\infty$$

6) $a_n = \frac{2n-3}{n+2}\sin(n)$ this is osillating, so the limit might not exist, or it exists if we can apply the squeeze th.

In this case it doesn't exist. To prove it we need to exhibit 2 subsequences with different limits.

For $n = k\pi$ $\quad a_{k\pi} = \frac{2k\pi - 3}{k\pi + 2}\sin(k\pi) = 0$ so $\lim_{k\to\infty} a_{k\pi} = 0$

for $n = \frac{\pi}{2} + 2k\pi$ $\quad a_{\frac{\pi}{2}+2k\pi} = \frac{\pi + 4k\pi - 3}{\pi/2 + 2k\pi + 2}\sin\left(\frac{\pi}{2} + 2k\pi\right)$

$$= \frac{4k\pi + \pi - 3}{2k\pi + \pi/2 + 2}$$

so $\lim_{k\to\infty} a_{\frac{\pi}{2}+2k\pi} = 2$

Since there 2 subsequences with different limits, $\lim_{n\to\infty} a_n$ doesn't exist.

7) $a_n = \dfrac{\sin(\frac{1}{n})}{n}$ Note that this sequence is not oscillating because $\lim_{n\to\infty} \sin(\frac{1}{n}) = \sin(0) = 0$

and the form "$\frac{0}{\infty}$" is not undefined, it is 0, so

$\lim_{n\to\infty} a_n = 0.$

Method 2 You can also use the squeeze theor.

$-1 \le \sin(\frac{1}{n}) \le 1$ so $\underset{\downarrow \atop 0}{-\frac{1}{n}} \le a_n \le \underset{\downarrow \atop 0}{\frac{1}{n}}$

so by squeeze theor $\lim_{n\to\infty} a_n = 0$

8) $a_n = n \sin(\frac{1}{n})$ Again, this is not oscillating, but the form "$\infty \cdot 0$" is undefined.

We try to write it as "$\frac{0}{0}$"

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \overset{H}{=} \lim_{n\to\infty} \frac{\cancel{-\frac{1}{n^2}} \cos(\frac{1}{n})}{\cancel{-\frac{1}{n^2}}} = \lim_{n\to\infty} \cos(\frac{1}{n}) = 1$$

Method 2 $\frac{1}{n} \to 0$ for $n \to \infty$ and we can use Taylor expansion

$$\sin(x) = x + o(x) \quad \text{so} \quad \sin(\tfrac{1}{n}) = \tfrac{1}{n} + o(\tfrac{1}{n})$$

$$\lim_{n\to\infty} n \sin(\tfrac{1}{n}) = \lim_{n\to\infty} n(\tfrac{1}{n} + o(\tfrac{1}{n})) = \lim_{n\to\infty} 1 + o(1) = 1.$$

9) $a_n = (-1)^n \dfrac{\sqrt{n}}{1-n}$ this is oscillating

$$-\frac{\sqrt{n}}{1-n} \le a_n \le \frac{\sqrt{n}}{1-n}$$
$\downarrow$ 0 $\qquad$ $\downarrow$ 0

So by squeeze theor $\lim_{n\to\infty} a_n = 0$

10) $a_n = (-1)^n \left(1 + \dfrac{2}{n}\right)^n$

$$a_{2k} = \left(1 + \frac{2}{2k}\right)^{2k} = \left[\left(1+\frac{1}{k}\right)^k\right]^2 \quad \text{so } \lim_{k\to\infty} a_{2k} = e^2$$

$$a_{2k+1} = -\left(1 + \frac{2}{2k+1}\right)^{2k+1} = -\left[\left(1+\frac{2}{2k+1}\right)^{\frac{2k+1}{2}}\right]^2$$

so $\lim_{k\to\infty} a_{2k+1} = -e^2$

Since there are 2 subsequences with different limits $\lim_{n\to\infty} a_n$ doesn't exist.

11) $a_n = \dfrac{n^2+1}{3n^3-n}$ $\qquad \lim_{n\to\infty} a_n = 0$

Long way: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{\cancel{n^2}(1+o(1))}{n^{\cancel{3}}(3+o(1))} = \lim_{n\to\infty} \dfrac{1+o(1)}{n(1+o(1))} = 0$

12) $a_n = \dfrac{\sqrt[3]{5n+1}}{\sqrt[6]{n^2+n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt[3]{n}\,\sqrt[3]{5+o(1)}}{\sqrt[6]{n^2}\,\sqrt[6]{1+o(1)}} = \lim_{n\to\infty} \frac{\cancel{\sqrt[3]{n}}}{\cancel{\sqrt[3]{n}}}\,\frac{\sqrt[3]{5+o(1)}}{\sqrt[6]{1+o(1)}} = \sqrt[3]{5}$$

PROBLEM 4

13) $\sum_{m=1}^{\infty} \frac{\sqrt{3m^5+m}}{m^2+\sqrt{m}}$ the series has positive terms, so absolute convergence and convergence are the same.

$$\left(\frac{\sqrt{3m^5+m}}{m^2+\sqrt{m}} \sim \frac{\sqrt{m^5}}{m^2} = \frac{m^{5/2}}{m^2} = \frac{1}{m^{2-5/2}} = \frac{1}{m^{-1/2}} = \sqrt{m} \right)$$

we already see that the term is not even going to 0

$$\lim_{m\to\infty} \frac{\sqrt{3m^5+m}}{m^2+\sqrt{m}} = \lim_{m\to\infty} \frac{m^{5/2}\sqrt{3+o(1)}}{m^2\,(1+o(1))} = \lim_{m\to\infty} \sqrt{m}\,\frac{\sqrt{3+o(1)}}{1+o(1)} = \infty$$

so the necessary condition for convergence fails, therefore the series is divergent.

(You can also use asymptotic comparison with the series $\sum \frac{1}{m^{-1/2}}$)

14) $\sum_{m=1}^{\infty} \frac{\sqrt[3]{m^6+3m}}{3m+m^3}$ positive terms so abs conv = conv.

$\left(\sim \frac{\sqrt[3]{m^6}}{m^3} = \frac{m^2}{m^3} = \frac{1}{m} \right)$ So we will use asymptotic comparison with $\sum \frac{1}{m}$

$$\lim_{m\to\infty} \frac{\frac{\sqrt[3]{m^6+3m}}{3m+m^3}}{\frac{1}{m}} = \lim_{m\to\infty} \frac{m\cdot\sqrt[3]{m^6+3m}}{3m+m^3} = \lim_{m\to\infty} \frac{\cancel{m}\cdot \cancel{m^2}\sqrt[3]{1+o(1)}}{\cancel{m^3}(o(1)+1)} = 1$$

and since $\sum \frac{1}{m}$ is divergent, by asymptotic comparison the series $\sum_{m=1}^{\infty} \frac{\sqrt[3]{m^6+3m}}{3m+m^3}$ is divergent.

15) $\sum_{n=1}^{\infty} \frac{\sqrt{\sqrt{n}+1}}{n\sqrt{n}}$ positive terms so abs conv. = conv.

$\left(\sim \frac{\sqrt{\sqrt{n}}}{n\sqrt{n}} = \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{3/2-1/4}} = \frac{1}{n^{5/4}} \right)$ so we try to apply asymptotic comparison

$$\lim_{n\to\infty} \frac{\frac{\sqrt{\sqrt{n}+1}}{n\sqrt{n}}}{\frac{1}{n^{5/4}}} = \lim_{n\to\infty} \frac{n^{5/4}\sqrt{\sqrt{n}}\,(1+o(1))}{n^{3/2}} = \lim_{n\to\infty} \frac{\cancel{n^{3/2}}(1+o(1))}{\cancel{n^{3/2}}} = 1$$

since $\sum \frac{1}{n^{5/4}}$ is convergent, by asymptotic comp the series (15) is convergent.

16) $\sum_{n=2}^{\infty} \frac{1}{n\log^3 n}$ We try to apply integral test

$f(x) = \frac{1}{x\log^3 x}$ is positive for $x > 1$
is continuous for $x > 1$

$$f'(x) = -(x\log^3 x)^{-2}\left(\log^3 x + x\,3\log^2 x \cdot \frac{1}{x}\right) = -(x\log^3 x)^{-2}(\log^3 x + 3\log^2 x)$$

$$= \boxed{-\frac{\log^2 x}{(x\log^3 x)^2}}(\log x + 3)$$

this is always negative

and $\log x + 3 > 0 \to \log x > -3 \to x > e^{-3}$ so $f'(x)$ is negative for $x > e^{-3} \to f(x)$ is decreasing for $x > e^{-3}$

Therefore we can apply the integral test

$$\int_2^{+\infty} \frac{1}{x\log^3 x}\,dx = \lim_{M\to\infty} \int_2^M \frac{1}{x\log^3 x}\,dx = \lim_{M\to\infty} -\frac{1}{2\log^2 x}\Big]_2^M$$

$$= \lim_{M\to+\infty}\left(-\frac{1}{2\log^2 M} + \frac{1}{2\log^2 2}\right) = \frac{1}{2\log^2 2} \qquad \text{convergent}$$

so by integral test the series is convergent.

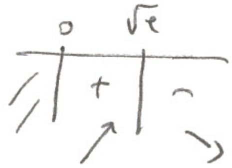
17) $\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ We try to use the integral test

$f(x) = \frac{\log x}{x^2}$ is positive for $x > 1$
continuous

$$f'(x) = \frac{\frac{1}{x}x^2 - 2x\log x}{x^2} = \frac{x(1-2\log x)}{x^2} = \frac{1-2\log x}{\boxed{x}} \to \text{always positive for natural numbers}$$

so we need to check ~~the~~ the sign of $1-2\log x$

$1-2\log x = 0 \to \log x = \frac{1}{2} \to x = \sqrt{e}$



$1-2\log(1) = 1 > 0$

$1-2\log(e) = -1 < 0$

so f is decreasing for $x > \sqrt{e}$

and we can apply the integral test

$$\int_1^{\infty} \frac{\log x}{x^2}\,dx = \lim_{M\to\infty}\int_1^M \frac{\log x}{x^2}\,dx$$

$$\int \frac{\log x}{x^2}\,dx = \int \frac{u\,du}{e^u} = \int e^{-u}u\,du = -e^{-u}u + \int e^{-u}\,du$$

$u = \log x \to x = e^u$

$du = \frac{dx}{x}$

$$= -e^{-u}u - e^{-u} + c$$

$$= -e^{-u}(u+1) + c$$

$$= -\frac{1}{x}(\log x + 1) + c$$

$$\lim_{M\to\infty} \int_1^M \frac{\log x}{x^2}\,dx = \lim_{M\to+\infty} -\frac{1}{x}(\log x+1)\Big]_1^M$$

$$= \lim_{M\to\infty} -\frac{1}{M}(\log M + 1) + 1 = 1 \qquad \left(\text{bec } \lim_{M\to\infty}\frac{\log M}{M} = 0 \text{ by L'Hospital} \right)$$

so by integral test the series is convergent.

18) $\sum_{m=1}^{\infty} \operatorname{arctg}(m)$ $\qquad \lim_{m\to\infty} \operatorname{arctg}(m) = \pi/2$

since the necessary condition for convergence fails, the series is divergent

19) $\sum_{m=1}^{\infty} (-1)^m \frac{2^m}{m!}$ $\qquad$ This is alternating so abs. conv $\neq$ conv.

check absolute conv. $\quad \sum_{m=1}^{\infty} \left| (-1)^m \frac{2^m}{m!} \right| = \sum_{m=1}^{\infty} \frac{2^m}{m!}$

Ratio Test $\quad \lim_{m\to\infty} \frac{\frac{2^{m+1}}{(m+1)!}}{\frac{2^m}{m!}} = \lim_{m\to\infty} \frac{\cancel{2^m}\cdot 2}{(m+1)\cdot \cancel{m!}} \cdot \frac{\cancel{m!}}{\cancel{2^m}} = \lim_{m\to\infty} \frac{2}{m+1} = 0 < 1$

so convergent by Ratio Test.

So the series (19) is absolutely convergent, and therefore also convergent.

20) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ Oscillating, so check abs. conv first.

Abs conv: $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{2n+1}$

$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$ Since $\sum \frac{1}{n}$ divergent

by asymptotic comparison $\sum \frac{1}{2n+1}$ is divergent.

Therefore the series (20) is NOT absolutely convergent.

check convergence $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{2n+1}$

$\frac{1}{2n+1}$ is decreasing to 0 so by oscillating series test

the series (20) is convergent.

21) $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{5n+6}$ Oscillating

Abs conv $\sum_{n=1}^{\infty} \left| (-1)^n \frac{3n-1}{5n+6} \right| = \sum_{n=1}^{\infty} \frac{3n-1}{5n+6}$ $\lim_{n\to\infty} \frac{3n-1}{5n+6} = 3/5$

so the necessary cond. for convergence fails, and the series is not absolutely convergent.

conv $a_n = (-1)^n \frac{3n-1}{5n+6}$

$a_{2k} = (-1)^{2k} \frac{6k-1}{10k+6}$ so $\lim_{k\to\infty} a_{2k} = 3/5$

$a_{2k+1} = (-1)^{2k+1} \frac{6k+2}{10k+11}$ so $\lim_{k\to\infty} a_{2k+1} = -3/5$

Since there are 2 subsequences with different limits $\lim_{n\to\infty} a_n$ doesn't exist, so the necessary condition fails and the series is not convergent.

22) $\sum_{n=1}^{\infty} \frac{n!}{2015^n}$ positive terms so conv = abs conv.

Ratio Test: $\lim_{n\to\infty} \frac{(n+1)!/2015^{n+1}}{n!/2015^n} = \lim_{n\to\infty} \frac{(n+1)\cdot \cancel{n!}}{\cancel{2015^n}\cdot 2015} \cdot \frac{\cancel{2015^n}}{\cancel{n!}}$

$= \lim_{n\to\infty} \frac{n+1}{2015} = +\infty > 1$ so divergent.

23) $\sum_{n=2}^{\infty} \left(\frac{2n^3+1}{n^3-1}\right)^n$ positive terms so conv = abs conv.

Root test $\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n^3+1}{n^3-1}\right)^n} = \lim_{n\to\infty} \frac{2n^3+1}{n^3-1} = 2 > 1$ so divergent

24) $\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^{n^2}$ positive terms so conv = abs conv

Root test $\cancel{\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n^3+1}{n^3-1}\right)^{n^2}} = \lim_{n\to\infty} 2n}$

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n-1}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n-1}{n}\right)^n = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n$

$= \lim_{n\to\infty} \left[\left(1+\frac{1}{-n}\right)^{-n}\right]^{-1} = e^{-1} < 1$ so convergent

25) $\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^n} = \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n \cdot \frac{1}{3} = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ positive terms

geometric series with base $3/2 > 1$ so divergent

26) $\sum_{n=1}^{\infty} \frac{4^{1-n}}{3} = \sum_{n=1}^{\infty} \frac{4}{3} \cdot \frac{1}{4^n} = \frac{4}{3} \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$ positive terms

geom. series with base $\frac{1}{4} < 1$ so convergent

27) $\sum_{n=1}^{\infty} \frac{(-5)^n}{2^{3n}} = \sum_{n=1}^{\infty} \left(-\frac{5}{8}\right)^n$

$\sum_{n=1}^{\infty} \left|\left(-\frac{5}{8}\right)^n\right| = \sum_{n=1}^{\infty} \left(\frac{5}{8}\right)^n$ geometric with base $\frac{5}{8} < 1$ so conv.

So the series (27) is absolutely conv, and therefore also convergent.

PROBLEM 5

28) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(-1)^n x^n}{4^n}$ centered at 0.

To find R we use the ratio test on the series with absolute values $\displaystyle\sum_{n=1}^{\infty} \left|\frac{n^2(-1)^n x^n}{4^n}\right| = \sum_{n=1}^{\infty} \frac{n^2|x|^n}{4^n}$

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2|x|^{n+1}}{4^{n+1}}}{\frac{n^2|x|^n}{4^n}} = \lim_{n\to\infty} \frac{(n+1)^2 \cancel{|x|^n}|x|}{\cancel{4^n}\cdot 4} \cdot \frac{\cancel{4^n}}{n^2\cancel{|x|^n}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{|x|}{4} = \frac{|x|}{4} < 1 \to |x| < 4 \quad \text{so } R=4.$$

To find I check convergence at the endpoints

$\underline{x=4}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(-1)^n 4^n}{4^n} = \sum_{n=1}^{\infty} n^2(-1)^n$ divergent because $\displaystyle\lim_{n\to\infty} n^2(-1)^n$ doesn't exist.

Indeed $a_{2k} = 4k^2(-1)^{2k} = 4k^2$ so $\lim a_{2k} = +\infty$

$a_{2k+1} = (4k^2+4k+1)(-1)^{2k+1}$

$= -(4k^2+4k+1)$ so $\lim a_{2k+1} = -\infty$

and since there are 2 subsequences with $\neq$ limits $\lim (-1)^n n^2$ doesn't exist.

$\underline{x=-4}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(-1)^n(-4)^n}{4^n} = \sum_{n=1}^{\infty} n^2$ divergent because $\displaystyle\lim_{n\to\infty} n^2 = \infty$ so the necessary cond fails

So $R=4$

$I = -4 < x < 4$

29) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\cdot x^n}{n^2}$ centered at 0

$$R: \lim_{n\to\infty} \frac{\dfrac{(n+1)!\,|x|^{n+1}}{(n+1)^2}}{\dfrac{n!\,|x|^n}{n^2}} = \lim_{n\to\infty} \frac{(n+1)\cdot \cancel{n!}\,\cancel{|x|^n}\,|x|}{(n+1)^2}\cdot\frac{n^2}{\cancel{n!}\,\cancel{|x|^n}}$$

$$= \lim_{n\to\infty} \frac{(n^3+n^2)\,|x|}{n^2+2n+1} = +\infty$$ so the series is divergent always, unless $x=0$

so $R=0$

$I:\ x=0$

30) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n x^n}{n!}$ center 0

$$R: \lim_{n\to\infty} \frac{e^{n+1}|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{e^n|x|^n} = \lim_{n\to\infty}\frac{e|x|}{n+1} = 0$$

so the series converges always

$R=\infty$

$I:\ \mathbb{R}$ (real numbers)

31) $\displaystyle\sum_{n=2}^{\infty} \frac{(x-1)^n}{n^2+1}$ centered at 1

R: $\displaystyle\lim_{n\to\infty} \frac{|x-1|^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{|x-1|^n} = \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2} \cdot |x-1| = |x-1| < 1$

So $R = 1$

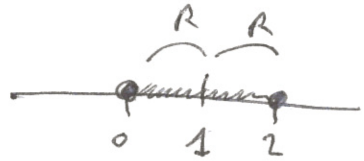
Endpoints: $|x-1| = 1 \to x-1 = \pm 1 \to x = 1 \pm 1$ $\begin{cases} x=0 \\ x=2 \end{cases}$

<u>$x=0$</u>: $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$ convergent by oscillating series test since $\frac{1}{n^2+1} \searrow 0$

<u>$x=2$</u>: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+1}$ convergent by asymptotic comp. with $\sum \frac{1}{n^2}$ (you can check $\lim \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = 1$)

So $R = 1$

I: $0 \le x \le 2$



32) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$ centered at 0

Here to find R is more convenient to use the root test

$$\lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^n |x|^n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)|x| = |x| < 1$$

so $R = 1$

<u>Endpoints</u> <u>$x = 1$</u> $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ divergent because

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$ so the necessary condition fails

<u>$x=-1$</u> $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n (-1)^n$ $a_n = \left(1+\frac{1}{n}\right)^n (-1)^n$

$a_{2k} = \left(1+\frac{1}{2k}\right)^{2k}$ so $\lim_{k\to\infty} a_{2k} = e$

$a_{2k+1} = \left(1+\frac{1}{2k+1}\right)^{2k+1} (-1)$ so $\lim_{k\to\infty} a_{2k+1} = -e$

Since there are 2 subseq. with $\neq$ limits, $\lim a_n$ doesn't exist, so the necessary cond. fails and the series is divergent

$R = 1$

$I: -1 < x < 1.$

PROBLEM 6

33) $$\frac{1}{10-x} = \frac{1}{10}\,\frac{1}{1-\frac{x}{10}} = \frac{1}{10}\,\frac{1}{1-(+\frac{x}{10})} = \frac{1}{10}\sum_{n=0}^{\infty}\left(+\frac{x}{10}\right)^n$$

for $\left|+\frac{x}{10}\right|<1$ so $|x|<10$ $\rightarrow$ $R=10$

$I: -10<x<10$

$$\frac{1}{10-x} = \sum_{n=0}^{\infty}\frac{x^n}{10^{n+1}}$$

34) $$\frac{x}{5+x^3} = \frac{x}{5}\,\frac{1}{1+\frac{x^3}{5}} = \frac{x}{5}\,\frac{1}{1-(-\frac{x^3}{5})} = \frac{x}{5}\sum_{n=0}^{\infty}\left(-\frac{x^3}{5}\right)^n$$

for $\left|-\frac{x^3}{5}\right|<1$ so $|x|^3<5$ $\rightarrow$ $|x|<\sqrt[3]{5}$ $\rightarrow$ $R=\sqrt[3]{5}$

$I: -\sqrt[3]{5}<x<\sqrt[3]{5}$

$$\frac{x}{5+x^3} = \frac{x}{5}\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{5^n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+1}}{5^{n+1}}$$

35) $$\frac{x^3}{(1-x)^2} = x^3\,\frac{1}{(1-x)^2} = x^3\,\frac{d}{dx}\left(\frac{1}{1-x}\right) = x^3\,\frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right)$$

for $|x|<1$ $\rightarrow$ $R=1$ $\quad I: -1<x<1$

$n-1=k$

$$= x^3\sum_{n=0}^{\infty}\frac{d}{dx}(x^n) = x^3\sum_{n=1}^{\infty}n\,x^{n-1} = x^3\sum_{k=0}^{\infty}(k+1)\,x^k = \sum_{k=0}^{\infty}(k+1)\,x^{k+3}$$

36) $$\operatorname{arctg}(4x) = \int\frac{4}{1+16x^2}\,dx = \int 4\sum_{n=0}^{\infty}(-16x^2)^n\,dx$$

$|16x^2|<1$ $\rightarrow$ $|x|<1/4$ $\quad R=1/4$

$$= \int 4\sum_{n=0}^{\infty}(-1)^n\,16^n\,x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n\,4^{2n+1}\int x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty}(-1)^n\,4^{2n+1}\,\frac{x^{2n+1}}{2n+1} + c$$

plugging $x=0$ we get $c=0$

you can check $I: -\frac{1}{4}\le x\le 1/4$

PROBLEM 7

37) $\sqrt[3]{x}$ $\quad x_0 = 1$

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$f(x) = \sqrt[3]{x}$ $\qquad f(1) = 1$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\qquad f'(1) = \frac{1}{3}$

$f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)x^{-5/3}$ $\qquad f''(1) = -\frac{2}{9}$

$f'''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)x^{-8/3}$ $\qquad f'''(1) = \frac{10}{27}$

$$P_3(x) = 1 + \frac{1}{3}(x-1) + \frac{1}{[illegible]}(x-1)^2 + \frac{10}{6 \cdot 27}(x-1)^3$$

38) $\sin x$ $\quad x_0 = \pi$

$f(x) = \sin x$ $\qquad f(\pi) = 0$

$f'(x) = \cos x$ $\qquad f'(\pi) = -1$

$f''(x) = -\sin x$ $\qquad f''(\pi) = 0$

$f'''(x) = -\cos x$ $\qquad f'''(\pi) = 1$

$$P_3(x) = -(x-\pi) + \frac{1}{6}(x-\pi)^3$$

PROBLEM 8

39) $y' = x e^{-y}$ separable variables

1) stationary sol: $e^{-y} = 0$ impossible

2) $\int \frac{dy}{e^{-y}} = \int x\,dx \rightarrow e^{y} = \frac{x^2}{2} + c \quad c \in \mathbb{R}$

$$\boxed{y = \ln\left(\frac{x^2}{2} + c\right) \quad c \in \mathbb{R}}$$

40) $y y' + x y^2 = 4x$

It's not linear because we have y^2, so let's try to use some algebra to see if we can separate variables.

$y = 0$ is not a solution, so we can divide by y

$$y' + xy = \frac{4x}{y}$$

$$y' = \frac{4x}{y} - xy$$

$$y' = x\left(\frac{4}{y} - y\right)$$

1) stationary sol: $\frac{4}{y} - y = 0 \rightarrow 4 = y^2 \rightarrow \boxed{y = \pm 2}$

2) $\int \frac{dy}{\frac{4}{y} - y} = \int x\,dx$

$$\int \frac{y}{4 - y^2}\,dy = \frac{x^2}{2} + c$$

$$-\frac{1}{2}\ln|4 - y^2| = \frac{x^2}{2} + c \qquad c \in \mathbb{R}$$

$\ln|4-y^2| = -x^2 + c \qquad c \in \mathbb{R}$

$|4-y^2| = e^{-x^2} \cdot e^{c} \qquad e^{c} = K > 0$

$4-y^2 = \pm K e^{-x^2} = \tilde{K} e^{-x^2} \qquad \tilde{K} \neq 0$

$-y^2 = -4 + \tilde{K} e^{-x^2}$

$y = \pm\sqrt{4 - \tilde{K} e^{-x^2}} \qquad \tilde{K} \neq 0$

General sol: $y = \pm\sqrt{4 - \tilde{K} e^{-x^2}} \qquad \tilde{K} \neq 0$

and $y = \pm 2$

Can be put together as $\boxed{y = \pm\sqrt{4 - c e^{-x^2}} \qquad c \in \mathbb{R}}$

41) $3y' + y = 2e^{-x}$ I order linear

$y' + \frac{1}{3} y = \frac{2}{3} e^{-x}$ Integrating factor $e^{\int \frac{1}{3} dx} = e^{\frac{1}{3}x}$

$y' \cdot e^{\frac{1}{3}x} + e^{\frac{1}{3}x} \frac{1}{3} y = \frac{2}{3} e^{-\frac{2}{3}x}$

$(y \cdot e^{x/3})' = \frac{2}{3} e^{-\frac{2}{3}x}$

$y \cdot e^{x/3} = \frac{2}{3} \int e^{-\frac{2}{3}x} dx = \frac{2}{3} e^{-\frac{2}{3}x} \left(-\frac{3}{2}\right) + c \qquad c \in \mathbb{R}$

$y = e^{-x/3} \left(-e^{-\frac{2}{3}x} + c\right) = -e^{-x} + c e^{-x/3} \qquad c \in \mathbb{R}$

General sol $\boxed{y(x) = -e^{-x} + c e^{-x/3} \qquad c \in \mathbb{R}}$

42) $y' - y\cos x = \dfrac{e^{\sin x}}{1+x^2}$ I order linear

Integrating factor $e^{\int -\cos x\,dx} = e^{-\sin x}$

$$y' \cdot e^{-\sin x} - y\cos x\, e^{-\sin x} = \frac{1}{1+x^2}$$

$$(y \cdot e^{-\sin x})' = \frac{1}{1+x^2}$$

$$y \cdot e^{-\sin x} = \operatorname{arctg}(x) + c \qquad c \in \mathbb{R}$$

$$\boxed{y(x) = e^{\sin x}\operatorname{arctg}(x) + c\,e^{\sin x} \qquad c \in \mathbb{R}}$$

43) $y'' - 2y' - 3y = e^{4x}$ II order, const coeff, non-homog, linear

1) solve the homogeneous $y'' - 2y' - 3y = 0$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda+1)(\lambda-3) = 0 \quad \begin{cases} \lambda = -1 \\ \lambda = 3 \end{cases}$$

general sol:

$$\boxed{y = c_1 e^{-x} + c_2 e^{3x} \qquad c_1, c_2 \in \mathbb{R}}$$

2) particular sol

4 is not a solution to the quadratic eq so look for a particular solution of the form $y = Ae^{4x}$

$y' = 4Ae^{4x}$ $y'' = 16Ae^{4x}$

$$16Ae^{4x} - 8Ae^{4x} - 3Ae^{4x} = e^{4x}$$

$$5Ae^{4x} = e^{4x} \quad \to \quad A = 1/5$$

General sol: $\boxed{y(x) = c_1 e^{-x} + c_2 e^{3x} + \frac{1}{5}e^{4x} \qquad c_1, c_2 \in \mathbb{R}}$

44) $y'' + y' = x - x^3$ II order, const coeff, linear, non homog.

1) homogeneous: $y'' + y' = 0 \quad \rightarrow \quad \lambda^2 + \lambda = 0 \quad \rightarrow \quad \lambda = 0, \ \lambda = -1$

$$\boxed{y(x) = c_1 + c_2 e^{-x} \qquad c_1, c_2 \in \mathbb{R}}$$

2) particular sol.

we are dealing with the poly $x - x^3 = (x - x^3) e^{0x}$

since 0 is a solution to the quadratic equation, look for

$$y = x(Ax^3 + Bx^2 + Cx + D) = Ax^4 + Bx^3 + Cx^2 + Dx$$

$$y' = 4Ax^3 + 3Bx^2 + 2Cx + D$$

$$y'' = 12Ax^2 + 6Bx + 2C$$

$$12Ax^2 + 6Bx + 2C + \underline{4Ax^3} + 3Bx^2 + 2Cx + D = x - x^3$$

$$4Ax^3 + (12A + 3B)x^2 + (6B + 2C)x + 2C + D = x - x^3$$

$$\begin{cases} 4A = -1 \\ 12A + 3B = 0 \\ 6B + 2C = 1 \\ 2C + D = 0 \end{cases} \qquad \begin{cases} A = -1/4 \\ B = -4A = 1 \\ C = \dfrac{1 - 6B}{2} = -5/2 \\ D = -2C = 5 \end{cases}$$

General sol: $y(x) = c_1 + c_2 e^{-x} - \frac{1}{4}x^4 + x^3 - \frac{5}{2}x^2 + 5x \qquad c_1, c_2 \in \mathbb{R}$

PROBLEM 9

45) $\begin{cases} y' = \dfrac{y^2-1}{xy} \\ y(1)=1 \end{cases}$ separable variables

1) stationary sol : $\dfrac{y^2-1}{y} = 0 \rightarrow \boxed{y = \pm 1}$

$\boxed{y=1}$ already solves the pb.

46) $\begin{cases} y' = 2\sqrt{y} \\ y(2) = 2 \end{cases}$ separable variables

1) stationary sol $\sqrt{y} = 0 \rightarrow \boxed{y=0}$

2) $\displaystyle\int \frac{dy}{2\sqrt{y}} = \int dx$

$\sqrt{y} = x + c \quad c \in \mathbb{R}$ we can square if $x+c \geq 0$

$y = (x+c)^2 \quad c \in \mathbb{R}$

$2 = (2+c)^2 \rightarrow c+2 = \pm\sqrt{2}$ choose the positive sol

$c+2 = \sqrt{2} \rightarrow c = \sqrt{2}-2$

sol: $\boxed{y(x) = (x+\sqrt{2}-2)^2}$

47) $\begin{cases} y' = y + x \\ y(1) = e - 2 \end{cases}$ I order linear

$y' - y = x$ integrating factor $e^{\int -1 dx} = e^{-x}$

$y' e^{-x} - y e^{-x} = x e^{-x}$

$(y \cdot e^{-x})' = x e^{-x}$

$y \cdot e^{-x} = \int x e^{-x} = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} + c \quad c \in \mathbb{R}$

$y = -x - 1 + c e^{x} \quad c \in \mathbb{R}$

$e - 2 = -1 - 1 + c e \rightarrow e = c e \rightarrow c = 1$

sol : $\boxed{y(x) = -x - 1 + e^{x}}$

48) $\begin{cases} y'' + 6y' + 9y = 0 \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$ II order linear, cost coeff, homog.

$\lambda^2 + 6\lambda + 9 = 0 \rightarrow (\lambda + 3)^2 = 0 \rightarrow \boxed{\lambda = -3}$

$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$ general sol $c_1, c_2 \in \mathbb{R}$

$y'(x) = -3 c_1 e^{-3x} + c_2 (e^{-3x} - 3x e^{-3x})$

$\begin{cases} y(0) = 1 \\ y'(0) = 2 \end{cases} \rightarrow \begin{cases} c_1 = 1 \\ -3c_1 + c_2 = 2 \end{cases} \quad \begin{cases} c_1 = 1 \\ c_2 = 5 \end{cases}$

Sol: $\boxed{y(x) = e^{-3x} + 5x e^{-3x}}$

49) $\begin{cases} y'' + 4y = x \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$ II order, const coeff, linear, non homog.

1) homogeneous $y'' + 4y = 0 \rightarrow \lambda^2 + 4 = 0 \rightarrow \lambda = \pm 2i$

$$\boxed{y(x) = c_1 \cos(2x) + c_2 \sin(2x) \qquad c_1, c_2 \in \mathbb{R}}$$

2) particular sol: 0 is not a solution to the quadr eq

→ look for solution of the type $y = Ax + B$

$y' = A$
$y'' = 0$ $\rightarrow 0 + 4A x + 4B = x \rightarrow \begin{cases} 4A = 1 \\ 4B = 0 \end{cases} \quad \begin{cases} A = 1/4 \\ B = 0 \end{cases}$

$$\boxed{\text{General sol: } \quad y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{4}x \qquad c_1, c_2 \in \mathbb{R}}$$

3) Initial conditions

$y'(x) = -2c_1 \sin(2x) + 2c_2 \cos(2x) + \frac{1}{4}$

$\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases} \quad \begin{cases} c_1 = 0 \\ 2c_2 + \frac{1}{4} = 1 \end{cases} \rightarrow \begin{cases} c_1 = 0 \\ c_2 = +3/8 \end{cases}$

$$\boxed{\text{solution: } \quad y(x) = +\frac{3}{8}\sin(2x) + \frac{1}{4}x}$$

PROBLEM

50) $\int e^{x+e^x}\,dx = \int e^x \cdot e^{e^x}\,dx$

$u = e^x$ $\qquad = \int e^u\,du = e^u + c = e^{e^x} + c$

$du = e^x\,dx$

51) $\int \sqrt{\dfrac{1-x}{1+x}}\,dx = \int \dfrac{\sqrt{1-x}}{\sqrt{1+x}} \cdot \dfrac{\sqrt{1-x}}{\sqrt{1-x}}\,dx = \int \dfrac{1-x}{\sqrt{1-x^2}}\,dx$

$= \int \dfrac{1}{\sqrt{1-x^2}}\,dx - \int \dfrac{x}{\sqrt{1-x^2}}\,dx = \arcsin x + \dfrac{1}{2}\int \dfrac{1}{\sqrt{u}}\,du$

$u = 1-x^2$

$du = -2x\,dx$

$= \arcsin x + \sqrt{1-x^2} + c$

52) $\int x\sqrt{1-x}\,dx$

Method 1 : $1-x = u^2 \quad \rightarrow \quad x = 1-u^2$

$-dx = 2u\,du$

$\int x\sqrt{1-x}\,dx = -\int (1-u^2)\,u\,2u\,du = -\int (2u^2 - 2u^4)\,du$

$= -\dfrac{2}{3}u^3 + \dfrac{2}{5}u^5 + c = -\dfrac{2}{3}(1-x)^{3/2} + \dfrac{2}{5}(1-x)^{5/2} + c$

Method 2 $\quad -\int -x\sqrt{1-x}\,dx = -\int (-1+1-x)\sqrt{1-x}\,dx$

$= -\int -\sqrt{1-x} + (1-x)\sqrt{1-x}\,dx = \int \sqrt{1-x}\,dx - \int (1-x)^{3/2}\,dx$

$= -(1-x)^{3/2} \cdot \dfrac{2}{3} + (1-x)^{5/2} \cdot \dfrac{2}{5} + c$

Method 3 by parts integrate $\sqrt{1-x}$ and diff x

$$\int x\sqrt{1-x}\,dx = -x\,(1-x)^{3/2}\cdot\frac{2}{3} + \int (1-x)^{3/2}\frac{2}{3}\,dx$$

$$= -\frac{2}{3}x(1-x)^{3/2} - (1-x)^{5/2}\cdot\frac{2}{5}\cdot\frac{2}{3} + C$$

(Note that this solution is the same as before

$$-x(1-x)^{3/2} = (-1+1-x)(1-x)^{3/2} = -(1-x)^{3/2} + (1-x)^{5/2}$$

So $-\frac{2}{3}x(1-x)^{3/2} - (1-x)^{5/2}\cdot\frac{2}{5}\cdot\frac{2}{3} = -\frac{2}{3}(1-x)^{3/2} + \frac{2}{3}(1-x)^{5/2} - (1-x)^{5/2}\frac{2}{5}\cdot\frac{2}{3}$

$$= -\frac{2}{3}(1-x)^{3/2} + \frac{2}{3}\left(1 - \frac{2}{5}\right)(1-x)^{5/2}$$

$$= -\frac{2}{3}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2}$$)